Similarly, we can show that

$$[\hat{L}_z, \hat{L}_-] = -\hbar \hat{L}_-$$

& $$[\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$$

(b) Commutation of $\hat{L}_x, \hat{L}_y, \hat{L}_z$ with $\hat{x}, \hat{y}, \hat{z}$.

$$[\hat{L}_x, \hat{x}] = (\hat{L}_x \hat{x} - \hat{x} \hat{L}_x)$$

$$= \left[y\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - z\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\right]$$

$$- x\left[y\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - z\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)\right]$$

If $\psi(x)$ is a function of x then

$$(\hat{L}_x \hat{x} - \hat{x}\hat{L}_x)\psi = \left[y\left(\frac{\hbar}{i}\frac{\partial}{\partial z}(x\psi)\right) - z\left(\frac{\hbar}{i}\frac{\partial}{\partial y}(x\psi)\right)\right]$$

$$- x\left[y\left(\frac{\hbar}{i}\frac{\partial \psi}{\partial z}\right) - z\left(\frac{\hbar}{i}\frac{\partial \psi}{\partial y}\right)\right]$$

$$= 0$$

$$\Rightarrow [\hat{L}_x, \hat{x}] = 0$$

Similarly, $[\hat{L}_x, \hat{y}] = i\hbar z$; $[\hat{L}_x, \hat{z}] = -i\hbar y$

$$[\hat{L}_y, \hat{y}] = 0 \quad , [\hat{L}_y, \hat{z}] = i\hbar x$$

$$[\hat{L}_y, \hat{x}] = -i\hbar z, [\hat{L}_z, \hat{y}] = -i\hbar x$$

$$[\hat{L}_z, \hat{z}] = 0, [\hat{L}_z, \hat{x}] = i\hbar y$$

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(c) Commutation of $\hat{L}$ with $\hat{p}$:

$$[\hat{L}_x, \hat{p}_x] = 0$$

$$[\hat{L}_x, \hat{p}_y] = i\hbar p_z$$

$$[\hat{L}_x, \hat{p}_z] = -i\hbar p_y$$

etc.

Dirac notation :-

Dirac notation reduces the labour of writing the integrals of wave function products.

For example if ψ_1 and ψ_2 be two wave function then,

$$\int \psi_1^* \psi_2 \, d\tau$$

is written as $\langle \psi_1 | \psi_2 \rangle$ and is called as scalar product of ψ_2 with ψ_1.

The condition of normalization of ψ_1 and ψ_2

$$\int \psi_1^* \psi_2 \, d\tau = 1$$

is written as

$$\langle \psi_1 | \psi_2 \rangle = 1$$

Similarly, the condition of orthonormality of ψ_m, ψ_n is

$$\langle \psi_m | \psi_n \rangle = \delta_{mn} \quad \text{where, } \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

If $\hat{A}$ is an Hermitian operator then in Dirac notation

$$\langle \psi_1 | \hat{A} \psi_2 \rangle = \langle \hat{A} \psi_1 | \psi_2 \rangle$$

also be written as

$$\langle \psi_1 | \hat{A} | \psi_2 \rangle$$

Properties of eigenfunctions and boundary condition :-

The expectation values $\langle x \rangle$ or $\langle p \rangle$ evaluated from a well-behaved wave function $\psi(x)$ turn out to be well behaved.

For this to happen $\psi(x)$ and $\frac{d\psi(x)}{dx}$ require to obey the following conditions

i) $\psi(x)$ and $\frac{d\psi(x)}{dx}$ must be single valued

ii) $\psi(x)$ and $\frac{d\psi(x)}{dx}$ must be finite.

iii) $\psi(x)$ and $\frac{d\psi(x)}{dx}$ must be continuous

If in a one-dimensional case, the particle lies betn x_1 and x_2 then the periodic boundary condition require

i) $\psi(x_1) = \psi(x_2)$

ii) $\left(\frac{\partial \psi}{\partial x}\right)_{x_1} = \left(\frac{\partial \psi}{\partial x}\right)_{x_2}$

Thes B.Cs are used in the analysis from mathematical view pt to obtain discrete energy values of a particle.

Probability current density :-

The time derivative of the position probability density ρ over a finite vol^m τ is

$$\frac{d}{dt}\int_\tau \rho(\vec{r},t)\,d\tau = \int_\tau \frac{\partial}{\partial t}(\psi^*\psi)\,d\tau$$

$$\Rightarrow \frac{dP}{dt} = \int_\tau \left(\psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right)d\tau$$

Now time-dependent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial \psi}{\partial t}$$

The complex conjugate of the above eqn^n is

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* = -i\hbar\frac{\partial \psi^*}{\partial t}$$

$$\therefore \frac{\partial \psi}{\partial t} = \frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right]$$

$$\& \ \frac{\partial \psi^*}{\partial t} = -\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*\right]$$

$$\therefore \psi\frac{\partial \psi^*}{\partial t} + \psi^*\frac{\partial \psi}{\partial t} = -\frac{i\hbar}{2m}\left[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right]$$

$$= -\frac{i\hbar}{2m}\nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi)$$

The probability current density is defined the probability of a particle passing through a unit cross-sectional area normal to its direction of motion per unit time.

$$\rightarrow \vec{J}(\vec{r},t) = \frac{i\hbar}{2m}\left[\psi\nabla\psi^* - \psi^*\nabla\psi\right]$$

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$$\therefore \int_\tau \left(\psi \frac{\partial \psi^*}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} \right) d\tau = - \int \vec{\nabla} \cdot \vec{J} \, d\tau$$

So the time rate of change of total probability P is

$$\frac{dP}{dt} = - \int_\tau \vec{\nabla} \cdot \vec{J} \, d\tau$$

$$= - \oint_S \vec{J} \cdot d\vec{s}$$ (from Gauss' divergence theorem)

$$\Rightarrow \frac{\partial}{\partial t} \int_\tau \psi^* \psi \, d\tau = - \int_\tau \vec{\nabla} \cdot \vec{J} \, d\tau$$

$$\Rightarrow \int_\tau \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right] d\tau = 0$$

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

→ Represents the conservation of probability density.